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Algebras of Type E_7 over Number Fields

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In [19], Tits described a method of constructing a Lie algebra of type E_7 over an arbitrary field k from an exceptional central simple Jordan algebra $\mathfrak{J}$ and a quaternion algebra $\mathfrak{A}$ defined over k. Using the well-known classification criteria of Cartan [3], he showed that this construction yields representatives of all isomorphism classes of real forms of E_7. In this paper, we obtain the same result for algebras of type E_7 over algebraic number fields (Theorem 4.3) making use of the Hasse principle for H^1 of a simply connected algebraic group of type E_7 [7] and the p-adic classification results of Kneser [11]. We obtain, in fact, a Hasse principle for H^1 of the adjoint group of type E_7 (Theorem 2.8), which yields necessary and sufficient conditions for isomorphism of two k-forms of E_7 in terms of the local behavior of the constituent Jordan algebra $\mathfrak{J}$ and the k-structure of $\mathfrak{A}$. In Section 5, we investigate the particular class of k-forms of E_7 that arise as derivation algebras of Freudenthal triple systems and apply the classification results of Section 4 to prove that every exceptional Freudenthal triple system over an algebraic number field is reduced. Finally, we apply the results of Section 5 to obtain classification of algebras of type $E_{6\text{II}}$ with split enveloping algebra over algebraic number fields, this last result having also been obtained by T. A. Springer by different methods in an unpublished work.

We assume throughout that char $k = 0$, though the results relating algebras of type E_7 to Freudenthal triple systems (Section 3, 5.2 and 5.3) remain valid for char $k \neq 2, 3$.

1

For an arbitrary field k with algebraic closure K, a Lie algebra $\mathfrak{L}$ defined over k is a k-form of E_7 (or of type E_7 over k) if $\mathfrak{L}_K = \mathfrak{L} \otimes_k K$ is of type E_7 in the classification of Killing–Cartan. Given such an $\mathfrak{L}$, one has a natural

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action of the Galois group $\Gamma(K/k)$ on $\mathfrak{L}_K$ via $s \to A(s) = I \otimes s$. Relative to this action, $\mathfrak{L}$ is precisely the set of $\Gamma(K/k)$ fixed elements $(\mathfrak{L}_K)^\Gamma$. If $\hat{\mathfrak{L}}$ is another k-form of E_7, $\hat{\mathfrak{L}} = (\hat{\mathfrak{L}}_K)^\Gamma$, there is a K-isomorphism $f\colon \mathfrak{L}_K \to \hat{\mathfrak{L}}_K$. $f^{-1}(\hat{\mathfrak{L}})$ is then a k-subalgebra of $\mathfrak{L}_K$ that is k-isomorphic to $\hat{\mathfrak{L}}$. $f^{-1}(\hat{\mathfrak{L}}) = (\mathfrak{L}_K)^\Gamma$ relative to the action $s \to f^{-1}(I \otimes s)f = B(s)$. It is well known [9] that there arises in this way a 1–1 correspondence between isomorphism classes of k-forms of E_7 and equivalence classes of $\Gamma(K/k)$-actions (s acting as an s-semiautomorphism) on $\mathfrak{L}_K$, where actions $s \to B(s)$, $s \to C(s)$ are equivalent if and only if there is an $A \in \operatorname{Aut} \mathfrak{L}_K$ such that $AB(s)A^{-1} = C(s)$ for all $s \in \Gamma$ (A induces a k-isomorphism from the $B(\Gamma)$-fixed points to the $C(\Gamma)$-fixed points). If one associates to each $\Gamma(K/k)$-action B the 1-cocycle $s \to B(s)A(s)^{-1}$, this 1–1 correspondence gives rise to the well-known result [16]: The isomorphism classes of k-forms of E_7 are in 1–1 correspondence with the elements of $H^1(\Gamma(K/k), \operatorname{Aut} \mathfrak{L}_K)$, where the $\Gamma(K/k)$-action on $\operatorname{Aut} \mathfrak{L}_K$ is given by ${}^s g = A(s)\,gA(s)^{-1}$.

The explicit correspondence, for later reference, is given by

Proposition 1.1. *Let $\mathfrak{L}$ be a k-form of E_7. Define $\Gamma(K/k)$-action on* $\operatorname{Aut} \mathfrak{L}_K$ *via $\mathfrak{L}$ as above. Then, $\gamma \in H^1(\Gamma(K/k)$,* $\operatorname{Aut} \mathfrak{L}_K)$ *corresponds to the set of fixed points of $\{\gamma(s)\, A(s) \mid s \in \Gamma(K/k)\}$.*

Henceforth, for the sake of simplicity, we shall denote by $H^i(k, G)$ the cohomology set $H^i(\Gamma(K/k), G)$. Also, as we have done already in Proposition 1.1, we shall write, for a cocycle γ, $\gamma \in H^i(k, G)$ while intending $\{\gamma\} \in H^i(k, G)$.

The correspondence of Proposition 1.1, while stated for k-forms of E_7, is, of course, valid for k-forms of any algebraic structure, a fact we shall exploit in Section 3.

2

Let $\mathfrak{L}$ be a k-form of E_7. Then, $\operatorname{Aut} \mathfrak{L}_K$ is an adjoint algebraic group of type E_7 that, with the $\Gamma(K/k)$-action of Section 1, is defined over k. Thus, there is a simply connected algebraic group of type E_7, defined over k, and a k-homomorphism p such that

$$1 \longrightarrow Z \xrightarrow{\;i\;} \tilde{G} \xrightarrow{\;p\;} \operatorname{Aut} \mathfrak{L}_K \longrightarrow 1 \tag{1}$$

is exact. Moreover, $Z = \text{center } \tilde{G}$ is cyclic of order 2, hence, it is a trivial $\Gamma(K/k)$-module. Setting $\bar{G} = \operatorname{Aut} \mathfrak{L}_K$ we have the cohomology sequence

$$H^0(k, \bar{G}) \xrightarrow{\;\delta\;} H^1(k, Z) \xrightarrow{\;i_1\;} H^1(k, \tilde{G}) \xrightarrow{\;p_1\;} H^1(k, \bar{G}) \xrightarrow{\;\Delta\;} H^2(k, Z), \tag{2}$$

which is exact in the sense of cohomology sets [17].

If $\gamma \in H^1(k, \bar{G})$, the entire sequence (2) can be "twisted" by γ to give a new exact sequence

$$H^0(k, {}_\gamma\bar{G}) \xrightarrow{{}_\gamma\delta} H^1(k, Z) \xrightarrow{{}_\gamma i_1} H^1(k, {}_\gamma\tilde{G}) \xrightarrow{{}_\gamma p_1} H^1(k, {}_\gamma\bar{G}) \xrightarrow{{}_\gamma\Delta} H^2(k, Z) \quad (3)$$

There is then a bijection τ_γ: $H^1(k, \bar{G}) \to H^1(k, {}_\gamma\bar{G})$ such that (i) $\tau_\gamma(\gamma) = 1$, and (ii) for $\beta \in H^1(k, \bar{G})$, $\Delta(\gamma) = \Delta(\beta)$ in (2) if and only if ${}_\gamma\Delta(\tau_\gamma(\beta)) = 1$, [16].

An immediate consequence of the exactness of (3) and the existence of τ_γ is

Proposition 2.1. *If $H^1(k, \tilde{G}) = 0$ for every simply connected algebraic group of type E_7 defined over k, Δ is injective.*

This is made particularly useful by

Theorem 2.2 (Kneser [11]). *If k is a p-adic field, then $H^1(k, \tilde{G}) = 0$ for every semisimple simply connected algebraic group defined over k.*

We thus have

Corollary 2.3. *If k is p-adic, Δ is injective.*

For the remainder of this section, k will be an algebraic number field, S will be the set of prime spots, and k_p will be the completion of k at p. We denote by S_∞ the set of $p \in S$ such that $k_p \cong \mathbb{R}$ and let t_p: $k_p \to \mathbb{R}$ be that isomorphism.

For any algebraic group G defined over k, we have the natural mappings $H^i(k, G) \to H^i(k_p, G)$, which make the following diagram (with exact rows) commute:

$$\begin{tikzcd}
H^1(k, Z) \arrow[r, "i_1"] \arrow[d, "a"] & H^1(k, \tilde{G}) \arrow[r, "p_1"] \arrow[d, "b"] & H^1(k, \bar{G}) \arrow[r, "\Delta"] \arrow[d, "c"] & H^2(k, Z) \arrow[d, "d"] \\
\prod_{p \in S_\infty} H^1(k_p, Z) \arrow[r] & \prod_{p \in S_\infty} H^1(k_p, \tilde{G}) \arrow[r] & \prod_{p \in S} H^1(k_p, \bar{G}) \arrow[r] & \prod_{p \in S} H^2(k_p, Z).
\end{tikzcd} \quad (4)$$

(We restrict to $p \in S_\infty$ in $\prod H^1(k_p, \tilde{G})$ as a consequence of Theorem 2.2.) Here $\bar{G}$ is an adjoint group of type E_7 defined over k, $\tilde{G}$ is a simply connected covering group.

Analyzing the mappings of (4) we have

Lemma 2.4. *d is injective.*

Proof. This is an immediate consequence of the interpretation of $H^2(k, \mathbb{Z}_2)$ as the classes of central associative division algebras of exponent $\leqslant 2$ [17] and the Hasse principle for such algebras [4].

LEMMA 2.5. *b is bijective.*

This is just the statement of the Hasse principle

THEOREM 2.6 (Harder [7]). *Let G be a simply connected linear algebraic group defined over k and containing no factor of type E_8. Then, $b: H^1(k, \tilde{G}) \to \prod_{p \in S_\infty} H^1(k_p, \tilde{G})$ is bijective.*

LEMMA 2.7. *a is surjective.*

Proof. For $p \in S_\infty$, $k_p \cong \mathbb{R}$, hence, $H^1(k_p, Z) \cong \mathbb{Z}_2$. For

$$w \in \prod_{p \in S_\infty} H^1(k_p, Z)$$

let $S_w^+ = \{p \in S_\infty \mid w(p) \text{ is the trivial homomorphism}\}$, $S_w^- = S_\infty - S_w^+$. Since there exist elements in k with arbitrarily prescribed signs at the real spots, we can select $a \in k$ such that $t_p(a) > 0$ if $p \in S_w^+$, $t_p(a) < 0$ if $p \in S_w^-$. If $w_0 \in \mathrm{Hom}(\Gamma(K/k), \mathbb{Z}_2) \cong H^1(k, Z)$ is given by $s(a^{1/2}) = w_0(s)\, a^{1/2}$, one sees easily that $a(w_0) = w$.

We can now prove

THEOREM 2.8. *Let $\bar{G}$ be an adjoint algebraic group of type E_7 defined over k. Then $c: H^1(k, \bar{G}) \to \prod_{p \in S} H^1(k_p, \bar{G})$ is injective.*

Proof. Suppose β, $\gamma \in H^1(k, \bar{G})$, $c(\gamma) = c(\beta)$. By Lemma 2.4 and the commutativity of (4), we have $\Delta(\gamma) = \Delta(\beta)$. Twisting the sequence by γ, we can assume $\gamma = 1$, $\Delta(\beta) = 1$ in (3). The result then follows from Lemmas 2.5 and 2.7, and a simple diagram chase.

If we consider the sequence (2) with $\bar{G} = \mathrm{Aut}(\mathfrak{L}_0)_K$, where $\mathfrak{L}_0$ is a split k-form of E_7 (hence, $\bar{G}$ is a Chevalley group over k), interpreting as above the elements of $H^1(k, \bar{G})$ as classes of k-forms of E_7, and those of $H^2(k, Z)$ as central simple associative k-algebras of exponent $\leqslant 2$, we can interpret Δ as assigning to each isomorphism class $\{\mathfrak{L}\}$ of k-forms of E_7, a class $\Delta\{\mathfrak{L}\}$ of associative division algebras. For ease of notation, we shall designate isomorphism classes by representatives. Thus, we have a natural mapping $\mathfrak{L} \to \Delta(\mathfrak{L})$ (if we were to take other than $\mathfrak{L}_0$ as defining the Galois action in (2) we would obtain a *different* mapping via Δ).

In this context we can rephrase Corollary 2.3 and Theorem 2.8:

COROLLARY 2.9. *If k is p-adic, $\mathfrak{L}$, $\hat{\mathfrak{L}}$ k-forms of E_7, then $\mathfrak{L} \cong \hat{\mathfrak{L}}$ if and only if $\Delta(\mathfrak{L}) = \Delta(\hat{\mathfrak{L}})$.*

THEOREM 2.10. *If k is an algebraic number field and $\mathfrak{L}$, $\hat{\mathfrak{L}}$ are k-forms of E_7, then $\mathfrak{L}_p \cong \hat{\mathfrak{L}}_p$ for all $p \in S$ implies $\mathfrak{L} \cong \hat{\mathfrak{L}}$ (where $\mathfrak{L}_p$ denotes $\mathfrak{L}_{k_p}$).*

3

To obtain specific information about the k-forms of E_7, we introduce in this section, an explicit realization of the groups $\tilde{G}$ and $\bar{G}$ in the sequence (1). We take $\mathfrak{J}$ to be the split exceptional central simple Jordan algebra over k with generic trace T, generic norm N, and denote by $\mathfrak{J}_0$ the space of elements of trace zero, by $x^{\#}$ the "adjoint" of x in $\mathfrak{J}$ [8]. Der $\mathfrak{J}$, the algebra of k-derivations of $\mathfrak{J}$, is a split simple Lie algebra of type F_4, and $\mathfrak{J}$, as Der $\mathfrak{J}$-module, decomposes as $k1 \oplus \mathfrak{J}_0$. We take $\mathfrak{M}(\mathfrak{J}) = \{(\xi, \eta, x, y) \mid \xi, \eta \in k, x, y \in \mathfrak{J}\}$ and define, for $X_i = (\xi_i, \eta_i, x_i, y_i)$

$$\langle X_1, X_2\rangle = \xi_1\eta_2 - \xi_2\eta_1 + T(x_1, y_2) - T(x_2, y_1),$$
$$q(X_1) = T(x_1{}^{\#}, y_1{}^{\#}) - \xi_1 N(x_1) - \eta_1 N(y_1) - \tfrac{1}{4}(T(x_1, y_1) - \xi_1\eta_1)^2.$$

Then, $\mathfrak{M}(\mathfrak{J})$ is a Freudenthal triple system relative to the ternary product $(X, Y, Z) \to XYZ$ such that $\langle X, YZW\rangle = q(X, Y, Z, W)$, the right side denoting the linearized version of the quartic form q [13].

The algebra of derivations, $\operatorname{Der}\mathfrak{M}(\mathfrak{J}) = \{D \in \operatorname{End}\mathfrak{M}(\mathfrak{J}) \mid D(XYZ) = D(X)YZ + XD(Y)Z + XYD(Z)\}$ is a split form of E_7 (it is in fact the algebra of endomorphisms preserving $\langle\ ,\ \rangle$ and q "infinitesimally"). $\operatorname{Aut}(\operatorname{Der}(\mathfrak{M}(\mathfrak{J}_K)))$ is thus the adjoint Chevalley group of type E_7 defined over k. $\operatorname{Aut}\mathfrak{M}(\mathfrak{J}_K)$ is a simply connected covering group for $\operatorname{Aut}(\operatorname{Der}(\mathfrak{M}(\mathfrak{J}_K)))$ via the mapping $T \to p(T), p(T)\colon D \to TDT^{-1}$ with kernel $\{\pm I\}$ [2].

Setting $\bar{G} = \operatorname{Aut}(\operatorname{Der}(\mathfrak{M}(\mathfrak{J}_K)))$, $\tilde{G} = \operatorname{Aut}\mathfrak{M}(\mathfrak{J}_K)$ in (2), we interpret, as in Section 1, $H^1(k, \tilde{G})$ as the set of isomorphism classes of k-forms of the Freudenthal triple system $\mathfrak{M}(\mathfrak{J}_K)$. We call such systems *exceptioanl* Freudenthal triple systems over k. Utilizing (2) and the explicit description of the k-form, cohomology class correspondence of Proposition 1.1, we have

LEMMA 3.1. *Let $\mathfrak{M}$ be an exceptional Freudenthal triple system over k, $\gamma \in H^1(k, \tilde{G})$ the corresponding cohomology class. Then,* Der $\mathfrak{M}$ *is the k-form of* $\operatorname{Der}(\mathfrak{M}(\mathfrak{J}_K))$ *with corresponding cohomology class $p_1(\gamma) \in H^1(k, \bar{G})$.*

The exactness of (2) thus yields

COROLLARY 3.2. *Let $\mathfrak{L}$ be a k-form of E_7. Then, $\Delta(\mathfrak{L}) = k$ if and only if there is an exceptional Freudenthal triple system $\mathfrak{M}$ such that $\mathfrak{L} \cong$* Der $\mathfrak{M}$.

In particular, for any exceptional central simple Jordan algebra $\hat{\mathfrak{J}}$ over k $\mathfrak{M}(\hat{\mathfrak{J}})$, defined as above, is a k-form of $\mathfrak{M}(\mathfrak{J}_K)$; hence, $\operatorname{Der}\mathfrak{M}(\hat{\mathfrak{J}})$ is a k-form of E_7 with $\Delta(\operatorname{Der}\mathfrak{M}(\hat{\mathfrak{J}})) = k$.

Our explicit realization of (2) also gives some additional information about the algebras $\Delta(\mathfrak{L})$, namely, PROPOSITION 3.3. *Let $\mathfrak{L}$ be a k-form of E_7. Then, index $\Delta(\mathfrak{L}) \mid 8$.*

Proof. Since exponent $\Delta(\mathfrak{L}) \leqslant 2$, ind $\Delta(\mathfrak{L}) = 2^l$, $l \geqslant 0$.
Considering the commutative diagram

$$\begin{tikzcd} H^1(k, \tilde{G}) \arrow[r, "p_1"] \arrow[d] & H^1(k, G) \arrow[r, "\Delta"] \arrow[d] & H^2(k, Z) \arrow[d] \\ H^1(k, SL(\mathfrak{M}(\mathfrak{J}_K))) \arrow[r] & H^1(k, PGL(\mathfrak{M}(\mathfrak{J}_K))) \arrow[r] & H^2(k, \hat{Z}), \end{tikzcd}$$

with vertical maps induced by the obvious group injections,

$$\hat{Z} = \text{center } SL(\mathfrak{M}(\mathfrak{J}_K)),$$

we see that for any $\mathfrak{L}$, $\Delta(\mathfrak{L})$ is the division algebra part of a form of End $\mathfrak{M}(\mathfrak{J}_K)$, hence, ind $\Delta(\mathfrak{L}) \mid \dim \mathfrak{M}(\mathfrak{J}_K) = 56$.

Of particular interest to us is the construction due to Tits [19]: let $\mathfrak{A}$ be a quaternion algebra over k, let $\hat{\mathfrak{J}}$ be an exceptional central simple Jordan algebra over k, and form the vector space $\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}}) = \mathfrak{A}_0 \otimes \hat{\mathfrak{J}} + \text{Der } \hat{\mathfrak{J}}$, where $\mathfrak{A}_0$ is the space of elements of trace zero in $\mathfrak{A}$. $\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}})$, with product

$$\begin{aligned}[]&[a \otimes x + D_1, b \otimes y + D_2] \\ &\quad = [a, b] \otimes xy - a \otimes D_2(x) + b \otimes D_1(y) + (2t(ab)[R_x, R_y] + [D_1, D_2]),\end{aligned}$$

($[a, b] = ab - ba$, t the trace form in $\mathfrak{A}$, $R_x: y \to y \cdot x$ in $\hat{\mathfrak{J}}$) is a simple Lie algebra. It has been shown in [8] that when $\mathfrak{A} \cong k_2$, $\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}}) \cong \text{Der } \mathfrak{M}(\hat{\mathfrak{J}})$. In particular, this implies $\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}})$ is a k-form of E_7. Corollary 3.2 yields one direction of

LEMMA 3.4. *$\Delta(\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}})) = k$ if and only if $\mathfrak{A} \cong k_2$.*

Proof. It remains to show that $\Delta\mathfrak{L}((\mathfrak{A}, \hat{\mathfrak{J}})) = k$ implies $\mathfrak{A}$ split. By Corollary 3.2, $\Delta(\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}})) = k$ implies $\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}}) \cong \text{Der } \mathfrak{M}$, $\mathfrak{M}$ an exceptional Freudenthal triple system over k. Thus, $\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}})$ has a 56-dimensional module $\mathfrak{M}$ over k. Extending the base field to K, we see that $\mathfrak{M}_K$ is the unique 56-dimensional module for the split algebra $\mathfrak{L}(\mathfrak{A}_K, \hat{\mathfrak{J}}_K)$. It follows from the results of [19], the characterization of the automorphism group of E_7 [15], and the specific form of the isomorphism $\mathfrak{L}(K_2, \hat{\mathfrak{J}}_K) \to \text{Der } \mathfrak{M}(\hat{\mathfrak{J}}_K)$ [8], that $\mathfrak{M}_K$, viewed as Der $\hat{\mathfrak{J}}_K$-module is a direct sum of a four-dimensional trivial module $\mathfrak{M}_K{}^0$, and two copies of irreducible modules isomorphic to $(\hat{\mathfrak{J}}_K)_0$. Moreover, $\mathfrak{M}_K{}^0$ is an irreducible $(\mathfrak{A}_K)_0 \otimes 1$-module. By standard descent arguments, this implies that the trivial component of $\mathfrak{M}$ as Der $\hat{\mathfrak{J}}$-module is a four-dimensional, absolutely irreducible $\mathfrak{A}_0 \otimes 1$-module. $\mathfrak{A}_0 \otimes 1$ is isomorphic as Lie algebra to the derived algebra of $\mathfrak{A}$, hence, it is of type A_1.

If $\mathfrak{A}$ is not split, $\mathfrak{A}_0$ does not admit a four-dimensional absolutely irreducible module (tensoring the regular representation of $(\mathfrak{A}_0)_K$ with the irreducible four-dimensional $(\mathfrak{A}_0)_K$-module yields a sum of three irreducible, inequivalent modules of dimensions 2, 4, and 6, hence, the same is true for the k-representations of $\mathfrak{A}_0$. But if $\mathfrak{A}_0$ has a two-dimensional nontrivial module, $\mathfrak{A}_0$, and hence, $\mathfrak{A}$, is split). Thus, $\mathfrak{A}$ must be split as desired. This yields immediately

PROPOSITION 3.5. *Let $\mathfrak{A}$ be a quaternion algebra over k, $\hat{\mathfrak{J}}$ as above. Then, $\Delta(\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}})) = \mathfrak{A}$ (identify k with k_2 for convenience).*

Proof. Suppose $\Delta(\mathfrak{L}(\mathfrak{A}, \hat{\mathfrak{J}})) = \mathfrak{B}$ and let $L \supseteq k$ be a generic splitting field for $\mathfrak{A}$. From the related cohomology sequences we see that $\mathfrak{B}_L = \Delta(\mathfrak{L}(\mathfrak{A}_L, \hat{\mathfrak{J}}_L))$, and since $\mathfrak{A}_L$ is split, $\mathfrak{B}_L \cong L_2$ by Lemma 3.4. By properties of generic splitting fields [14], either $\mathfrak{B} \cong \mathfrak{A}$ or $\mathfrak{B} \cong k_2$. If $\mathfrak{B} \cong k_2$, Lemma 3.4 implies $\mathfrak{A} \cong k_2$, so the result follows.

The result 3.5 has been proved by Tits in [18] in a more general setting. We include a proof here because of its very simple nature.

4

With the results of Sections 2 and 3, we now can treat easily the classification questions for k-forms of E_7 for k real, p-adic, or an algebraic number field. The case k a finite field is well known, the only k-form of E_7 is the split form which is isomorphic to $\mathfrak{L}(k_2, \mathfrak{J})$, $\mathfrak{J}$ split.

For k p-adic we have

THEOREM 4.1. *Let k be p-adic, $\mathfrak{L}$ of type E_7 over k. Then, $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{A}, \mathfrak{J})$ for some $\mathfrak{A}$, $\mathfrak{J}$ as in Section 3. Moreover, $\mathfrak{L}(\mathfrak{A}, \mathfrak{J}) \cong \mathfrak{L}(\mathfrak{A}', \mathfrak{J}')$, if and only if $\mathfrak{A} \cong \mathfrak{A}'$.*

Proof. By Corollary 2.9, Δ is injective into the classes of k-algebras of exponent $\leqslant 2$. Since these are precisely the quaternion algebras, the $\mathfrak{L}(\mathfrak{A}, \mathfrak{J})$ exhaust all possibilities. The sufficiency of $\mathfrak{A} \cong \mathfrak{A}'$ follows since all p-adic $\mathfrak{J}$ are split.

For the real case we invoke the results of Tits [19], which are analyzed in detail in [8].

THEOREM 4.2. *Let $\mathfrak{L}$ be an $\mathbb{R}$-form of E_7, then, $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{A}, \mathfrak{J})$ for some $\mathfrak{A}$, $\mathfrak{J}$, as in Section 3. Moreover, any two real forms are isomorphic if and only if they have the same signature for their Killing forms.*

Recalling that any real $\mathfrak{J}$ is isomorphic to an algebra $\mathfrak{H}(\mathfrak{O}_3, \gamma)$, symmetric 3×3 matrices with entries in a composition octonion algebra $\mathfrak{O}$ of type

$(-1, -1, \mu)$ relative to conjugation by $\gamma = \text{diag}\{1, 1, \rho\}$ [1], we will be able to analyze isomorphism of forms if we can compute signatures for $\mathfrak{L}(\mathfrak{A}, \mathfrak{H}(\mathfrak{O}_3, \gamma))$. From [8], we obtain the following table of signatures

	$\mu > 0, \rho > 0$	$\mu > 0, \rho < 0$	$\mu < 0, \rho > 0$	$\mu < 0, \rho < 0$
$\mathfrak{A}$ split	7	7	-25	-25
$\mathfrak{A}$ compact	-5	-5	-133	-5

(5)

Clearly, by Proposition 3.5, $\mathfrak{A} \cong \mathfrak{A}'$ is a necessary condition for $\mathfrak{L}(\mathfrak{A}, \mathfrak{J}) \cong \mathfrak{L}(\mathfrak{A}', \mathfrak{J}')$.

Finally, for number fields we have

THEOREM 4.3. *Let k be an algebraic number field, let $\mathfrak{L}$ be a k-form of E_7. Then, $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{A}, \mathfrak{J})$ for some $\mathfrak{A}, \mathfrak{J}$, as in Section* 3.

Proof. By Theorem 2.10, it suffices to find $\mathfrak{A}, \mathfrak{J}$ such that for all $p \in S$, $\mathfrak{L}(\mathfrak{A}_p, \mathfrak{J}_p) \cong \mathfrak{L}_p$. Since k is a number field, exponent $\Delta(\mathfrak{L}) \leqslant 2$ implies $\Delta(\mathfrak{L})$ is a quaternion algebra. Setting $\mathfrak{A} = \Delta(\mathfrak{L})$, we see that for any $\mathfrak{J}$, and any finite p, $\mathfrak{L}(\mathfrak{A}_p, \mathfrak{J}_p) \cong \mathfrak{L}_p$ since $\Delta(\mathfrak{A}_p, \mathfrak{J}_p) = \mathfrak{A}_p$ by Proposition 3.5, and $\Delta(\mathfrak{L})_p = \Delta(\mathfrak{L}_p)$ (using Theorem 4.1). The same is true for all p such that $k_p \cong \mathbb{C}$, so we need only look at $p \in S_\infty$. Let $p_1, ..., p_k \in S_\infty$ be such that $\mathfrak{A}_{p_i}$ is split, $p_{k+1}, ..., p_n$ such that $\mathfrak{A}_{p_i}$ is compact. For each i, select a real octonion algebra $\mathfrak{O}^i = (-1, -1, \mu_i)$, and a $\gamma^i = \text{diag}\{1, 1, \rho_i\}$ so that $\mathfrak{L}_{p_i} \cong \mathfrak{L}(\mathfrak{A}_{p_i}, \mathfrak{H}(\mathfrak{O}_3{}^i, \gamma^i))$. Pick $\mu \in k$ such that $t_{p_i}(\mu) > 0$ if $\mu_i > 0$, $t_{p_i}(\mu) < 0$ if $\mu_i < 0$ for $1 \leqslant i \leqslant k$; $t_{p_i}(\mu) < 0$ for $k+1 \leqslant i \leqslant n$. Similarly, pick $\rho \in k$ such that $t_{p_i}(\rho)$ is arbitrary for $1 \leqslant i \leqslant k$; $t_{p_i}(\rho) > 0$ if $\mu_i < 0, \rho_i > 0$, $t_{p_i}(\rho) < 0$ otherwise for $k+1 \leqslant i \leqslant n$. Using (5), we see that $\mathfrak{L}(\mathfrak{A}, \mathfrak{H}(\mathfrak{O}_3, \gamma))_p \cong \mathfrak{L}_p$ for all $p \in S$, where $\mathfrak{O} = (-1, -1, \mu)$, $\gamma = \text{diag}\{1, 1, \rho\}$.

Since every exceptional central simple Jordan algebra over k is isomorphic to some $\mathfrak{H}(\mathfrak{O}_3, \gamma)$, $\mathfrak{O} = (-1, -1, \mu)$, $\gamma = \text{diag}\{1, 1, \rho\}$, [1], we have in fact, using the local information of Theorems 4.1 and 4.2, complete classification of k-forms of E_7, k a number field. Necessary and sufficient conditions for isomorphism of $\mathfrak{L}(\mathfrak{A}, \mathfrak{J})$ and $\mathfrak{L}(\mathfrak{A}', \mathfrak{J}')$, involving only properties of $\mathfrak{J}, \mathfrak{J}', \mathfrak{A}, \mathfrak{A}'$ over k (not directly using local results) are not known.

5

The E_7, exceptional Freudenthal triple system connection exploited in Section 3 in studying k-forms of E_7, also can be used, in conjunction with the classification results of Sections 3 and 4 to obtain structural information about

Freudenthal triple systems over special fields. We recall that a Freudenthal triple system $\mathfrak{M}$ is *reduced* if $\mathfrak{M}$ contains a strictly regular element u. By [6], $\mathfrak{M}$ is reduced if and only if $\mathfrak{M} \cong \mathfrak{M}(\mathfrak{J})$ for some exceptional central simple Jordan algebra over k. Our main result here will be

THEOREM 5.1. *Let $\mathfrak{M}$ be an exceptional Freudenthal triple system over k, k real, p-adic, or an algebraic number field. Then, $\mathfrak{M}$ is reduced.*

For the proof, we investigate more closely the $\mathfrak{M} \leftrightarrow \operatorname{Der} \mathfrak{M}$ correspondence, obtaining first

LEMMA 5.2. *Let $\mathfrak{M}, \mathfrak{M}'$ be exceptional Freudenthal triple systems over k. Then,* $\operatorname{Der} \mathfrak{M} \cong \operatorname{Der} \mathfrak{M}'$ *if and only if there is* $\varphi' \in \operatorname{Hom}(\mathfrak{M}, \mathfrak{M}')$ *such that* $q'(\varphi'(m)) = \lambda' q(m)$ *for all* $m \in \mathfrak{M}$, *some fixed* $\lambda' \in k^*$.

Proof. Consider the sequence (2) with $\tilde{G} = \operatorname{Aut} \mathfrak{M}(\mathfrak{J}_K)$, as in Section 3. If $\mathfrak{M}$ corresponds to $\gamma \in H^1(k, \tilde{G})$, we twist the sequence by γ and assume, in (3), that $\mathfrak{M}$ corresponds to $1 \in H^1(k, {}_\gamma\tilde{G})$, $\mathfrak{M}'$ corresponds to τ. Then, $\operatorname{Der} \mathfrak{M} \cong \operatorname{Der} \mathfrak{M}'$ is equivalent, by Lemma 1, to ${}_\gamma p_1(\tau) = 1$. Exactness of (3) yields $\tau = {}_\gamma i_1(\xi)$, $\xi \in H^1(k, \{\pm 1\})$. Since $\xi^{-1} \in H^1(k, K^*) = \{1\}$ (Hilbert's theorem 90), there is $\lambda \in K^*$ such that $\xi_s^{-1} = (\lambda^{-1})^s \lambda$. ${}_\gamma i_1(\xi) = \tau$ implies there is $\varphi \in \operatorname{Aut} \mathfrak{M}(\mathfrak{J}_K)$ such that $\varphi^{-1} \tau_s{}^s \varphi = \xi_s$ for all $s \in \Gamma(K/k)$, hence, $(\varphi\lambda)^{-1} \tau_s{}^s(\varphi\lambda) = 1$. Interpreting this in $H^1(k, {}_\gamma GL(\mathfrak{M}(\mathfrak{J}_k)))$, we see that $\varphi' = \varphi\lambda$ is a k-linear transformation of $\mathfrak{M}$ onto $\mathfrak{M}'$, and since φ preserves the quartic form on $\mathfrak{M}(\mathfrak{J}_K)$, φ' satisfies the condition of the lemma.

The reverse direction is clear since conjugation by φ' yields the desired isomorphism.

If $\mathfrak{M}$ in Lemma 5.2 is reduced, hence, $\mathfrak{M} \cong \mathfrak{M}(\mathfrak{J})$ for some $\mathfrak{J}$, the proof of Corollary 7.3 of [6] can be modified to show that $\mathfrak{M}'$ is also reduced, that $\lambda' = \rho^2$, $\rho \in k^*$, and that $\langle \varphi'(m_1), \varphi'(m_2) \rangle' = \pm\rho \langle m_1, m_2 \rangle$ for all $m_i \in \mathfrak{M}$. Moreover, the transformation $T = I_\pi{}^\rho$ given by (15) of [6] satisfies $q(T(m_1)) = \rho^2 q(m_1)$, $\langle T(m_1), T(m_2) \rangle = \pm\rho \langle m_1, m_2 \rangle$, the sign depending on whether $\pi = (12)$, or $\pi = (1)$. Thus, $\varphi' T^{-1} \colon \mathfrak{M} \to \mathfrak{M}'$ preserves both the quartic form and the skew bilinear form, and hence, is an isomorphism, so we have

COROLLARY 5.3. *Let $\mathfrak{M}, \mathfrak{M}'$ be as in Lemma 5.2, $\mathfrak{M}$ reduced. Then,* $\operatorname{Der} \mathfrak{M} \cong \operatorname{Der} \mathfrak{M}'$ *if and only if* $\mathfrak{M} \cong \mathfrak{M}'$.

The proof of Theorem 5.1 is now immediate, since if $\mathfrak{M}$ is an exceptional Freudenthal triple system, $\operatorname{Der} \mathfrak{M}$ is a k-form of E_7 that satisfies $\Delta(\mathfrak{L}) = k$ by Corollary 3.2. By Theorem 4.1, 4.2, or 4.3, we have $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{A}', \mathfrak{J}')$, and by Lemma 3.4, $\mathfrak{A}' \cong k_2$. The results of [8] then imply $\mathfrak{L} \cong \operatorname{Der} \mathfrak{M}(\mathfrak{J}')$ for some exceptional central simple Jordan algebra $\mathfrak{J}'$. Thus, $\operatorname{Der} \mathfrak{M} \cong \operatorname{Der} \mathfrak{M}(\mathfrak{J}')$, and since $\mathfrak{M}(\mathfrak{J}')$ is reduced, so is $\mathfrak{M}$ by Corollary 5.3.

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In the light of the results of [5] and [6], we can utilize Theorem 5.1 to classify Lie algebras of type E_{6II} with split envelope, showing in particular

THEOREM 6.1. *Let $\mathfrak{L}$ be of type E_{6II} over a real, p-adic, or algebraic number field k. If $\mathfrak{L}^*$ (defined as in* [5]) $\cong L_{27}$, $L = k(\lambda^{1/2})$, *then* $\mathfrak{L} \cong \mathfrak{L}(\mathfrak{J})_\lambda = \{\lambda^{1/2}R_a + D \mid a \in \mathfrak{J}_0, D \in \operatorname{Der} \mathfrak{J}\}$ *for some exceptional central simple Jordan algebra $\mathfrak{J}$ over k.*

This will extend Theorem 5 of [5] to include algebraic number fields with the additional hypothesis $\mathfrak{L}^*$ split.

The results of Section 5 are necessary to prove

LEMMA 6.2. *Let $\mathfrak{J}$ be exceptional central simple Jordan over k, k, L as in Theorem* 6.1, *and let s denote the nontrivial k-automorphism of L. Let C be an s-semilinear transformation of $\mathfrak{J}$ satisfying* (i) $N(C(x)) = \rho N(x)^s$ *for all* $x \in \mathfrak{J}$, *some* $\rho \in k^*$, *and* (ii) $C^t = C$, *t denoting transpose relative to* $T(\ ,\ )$ *in* $\mathfrak{J}$. *Then, there is* $y \in \mathfrak{J}$ *such that* $C(y) = \mu y^{\#}$, $\mu \in L^*$, $N(y) \neq 0$.

Proof. Consider the transformation $S = C_{(12)}^{-1}$ of [6] acting in $\mathfrak{M}(\mathfrak{J})$. S is clearly an s-semiautomorphism of $\mathfrak{M}(\mathfrak{J})$, and by (ii), $S^2 = I$. The fixed points of S in $\mathfrak{M}(\mathfrak{J})$ are thus a k-form of $\mathfrak{M}(\mathfrak{J}_K)$, hence, they are an exceptional Freudenthal triple system $\mathfrak{M}$ over k. By Theorem 5.1, $\mathfrak{M}$ is reduced and the Lemma follows immediately from [6, Lemma 8.1].

Turning now to the proof of Theorem 6.1, we recall first that under the assumptions of the theorem, $\mathfrak{L}_L \cong \mathfrak{L}(\hat{\mathfrak{J}}) = \{R_a + D \mid T(a) = 0, D \in \operatorname{Der} \hat{\mathfrak{J}}\}$ [5]. Moreover, $\hat{\mathfrak{J}}$ can be replaced by any isotope $\hat{\mathfrak{J}}^{(x)}$ [8]. Using an argument analogous to that preceding [5, Lemma 3] (which depends only on $\hat{\mathfrak{J}}$ being reduced) we may assume $\mathfrak{L} = \{l \in \mathfrak{L}(\mathfrak{J}) \mid l = -C^{-1}l^tC\}$, where C satisfies the assumptions of Lemma 6.2. By the Lemma, there is invertible $y \in \hat{\mathfrak{J}}$ such that $C(y^{-1}) = \mu\ (y^{-1})^{\#}$, $\mu \in L^*$. Since in $\hat{\mathfrak{J}}^{(y)}$, the trace bilinear form is $T^{(y)}(v, w) = T(U_y(v), w)$ [12], the transpose relative to $T^{(y)}$ is $X^{t_y} = U_y^{-1}X^tU_y$. Setting $D = U_y^{-1}C$, we have, thus,

$$\mathfrak{L} = \{l \in \mathfrak{L}(\hat{\mathfrak{J}}^{(y)}) \mid l = -D^{-1}l^{t_y}D\},$$

where $D(y^{-1}) = U_y^{-1}C(y^{-1}) = \mu U_{y^{-1}}((y^{-1})^{\#}) = \mu N(y^{-1})\, U_{y^{-1}}(y) = \mu N(y^{-1})y^{-1}$, and $D^{t_y} = D$. By results of [10, p. 244], D is thus a scalar multiple of an s-semiautomorphism A of $\hat{\mathfrak{J}}^{(y)}$, which is self adjoint, hence, of order 2. The result follows as in [5, Theorem 3, Corollary].

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